Second Law of Nonequilibrium Thermodynamics at Different Levels of Description

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In the case where the dynamics of a system may be described by two or more stochastic processes at different scales, the total entropy production, the sum of Shannon entropy increment in the system and entropy production due to heat transfer, may depend on the levels of description. By analyzing general Markov stochastic processes with a parameter representing the extent of separation of time scales, we find that the excess entropy production, a key quantity in steady state thermodynamics, is invariant with respect to the change in the level of description. We demonstrate our findings in a two-dimensional overdamped Langevin model where the total entropy production differs between the levels. These results suggest that the second law-like relations involving the excess entropy production are level-independent relations even in the nonequilibrium setup, and therefore act as promising alternatives to the ordinary second law.

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Introduction.— Since Einstein's study on Brownian motion [1], the applicability of thermodynamics has been extended to fluctuating worlds. In the framework of small systems thermodynamics, several thermodynamic quantities such as heat, work and entropy are naturally defined, and it can be verified that they satisfy the mesoscopic version of thermodynamic relations [2]. Recent development of experimental techniques enable the measurement of fluctuating quantities in various setups including biological molecular motors [3], mesoscopic quantum systems [4] and information processings [5], which makes more important the interpretation of measurable quantities from the thermodynamic point of view.

In the mesoscopic level, a system is often modeled in various ways where the dynamics is described by different sets of variables corresponding to which subsystem is focused on. Accordingly, heat, which is essentially the energy transfer to the hidden degrees of freedom [6], changes its value depending on the levels of description. Sekimoto discussed the relation between the heat at different levels, for a mesoscopic system that consists of a slow variable x, a fast variable y, and the environment [7]. It was pointed out that the relation between the two heat, $Q^{x,y}$ and Q^x, corresponding to the (x, y)-level and the x-level descriptions, respectively, is analogous to that between the effective Hamiltonians in the equilibrium statistical mechanics. It follows from the analogy that the total entropy production, which is the sum of the Shannon entropy increment in the system of interest and the heat divided by temperature of the environment, has level-independence and *objectivity* in the sense defined in [7]. The physical meaning of this independence is that the difference in the heat, $Q^{x,y} - Q^x$, amounts to the heat exchanged reversibly with the environment (reversible heat). The above statement follows the intuition that when the fast variable y is thermalized in the slow time scale of the x-dynamics, y could be regarded as a part of the environment.

In contrast to the preceding argument, it has been shown that in a certain non-equilibrium setup, the total entropy production may change its value depending on the levels of description, even when the separation of time scale justifies the elimination of the fast variables from the dynamics [8]. Such difference in the total entropy production, which we call the *hidden entropy production* [9], is expected to be commonly observed in non-equilibrium systems, implying that in a general setup, the physical laws involving the entropy production, such as the second law of thermodynamics, is not invariant with respect to the change in the levels of description. It is necessary for solving this problem to describe the physical laws by using quantities that do not depend on the levels.

In the present Letter, we focus on the excess part of entropy production [10, 11] which is a key quantity to construct steady state thermodynamics (SST), a possible extension of thermodynamics to nonequilibrium steady state. We show for general Markov processes and several definitions of excess entropy productions [12–14] that the sum of the system's entropy increment and the excess entropy production is kept invariant between the different levels of description, even when the hidden entropy production exists. These invariances guarantee the objectivity of second laws of nonequilibrium thermodynamics.

Invariance of excess entropy production.— Let us consider a general Markov process following the master equation,

$$\frac{\partial P_t(x,y)}{\partial t} = \mathcal{L}^{x,y}_{\lambda_t} \cdot P_t(x,y). \tag{1}$$

Here, P_t is the probability density function, x and y represent slow and fast variables, respectively, $\mathcal{L}^{x,y}_{\lambda_t}$ is the generator of time evolution which depends on time-dependent external parameters denoted by λ_t. By integrating (1) with respect to y, we may formally write the reduced master equation as

$$\frac{\partial P_t(x)}{\partial t} = \mathcal{L}^{x}_{\lambda_t} \cdot P_t(x). \tag{2}$$

Equation (2) gives the closed Markov dynamics of x if the reduced generator of time evolution, $\mathcal{L}^x_{\lambda_t} = \int \mathcal{L}^{x,y}_{\lambda_t} \cdot \frac{P_t(x,y)}{P_t(x)} \mathrm{d}y$, depends only on λ_t and not explicitly on t. Denoting the time scale of the x, y dynamics and λ_t modulation as τ_x, τ_y and τ_λ, respectively, this condition is satisfied when the parameter of separation of time scale, $\eta := \max\{\tau_y/\tau_x, \tau_y/\tau_\lambda\}$, is sufficiently

small, and the probability density for y relaxes to the stationary distribution under given x and λ_t,

$$P_t(x,y) = P^{\lambda_t}(y|x)P_t(x) + O(\eta). \tag{3}$$

In several systems modeled as Markov processes [15, 16], it has been considered that the entropy production due to the energy transfer to the hidden degrees of freedom is given by the transition probability $W^{\lambda_t}_{\Delta t}(\mathbf{x}'|\mathbf{x})$ from $\mathbf{x}$ to $\mathbf{x}'$ between time t and $t+\Delta t$ as,

$$\sigma(t,\lambda_t) := \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\langle \ln\frac{W^{\lambda_t}_{\Delta t}(\mathbf{x}_{t+\Delta t}|\mathbf{x}_t)}{W^{\bar{\lambda}_t}_{\Delta t}(\bar{\mathbf{x}}_t|\bar{\mathbf{x}}_{t+\Delta t})}\right\rangle, \tag{4}$$

at both levels of description, $\mathbf{x} = (x,y)$ and x. Here, $\langle\cdot\rangle$ represents the average over $\mathbf{x}_t$ and $\mathbf{x}_{t+\Delta t}$ at times t and $t+\Delta t$, $\bar{\mathbf{x}}$ indicates the time-reversal of $\mathbf{x}$, and the Boltzmann constant is set to unity. We call this entropy production with a prefix "ordinary" to avoid ambiguity. To take into account the reversible heat, we define the Shannon entropy of $\mathbf{x}$ as $S[P_t(\mathbf{x})] := -\int P_t(\mathbf{x})\ln P_t(\mathbf{x})\mathrm{d}\mathbf{x}$, and consider the total entropy production,

$$\Sigma(t,\lambda_t) := \dot{S}[P_t(\mathbf{x})] + \sigma(t,\lambda_t) \geq 0. \tag{5}$$

Here, $\dot{}$ indicates the time derivative. Inequality (5) holds for an arbitrary $P_t(\mathbf{x})$ and shall be interpreted as the mesoscopic version of the second law. We later show through an explicit example that in certain cases not only the entropy production, $\sigma(t,\lambda_t)$, but also the total entropy production, $\Sigma(t,\lambda_t)$, may have different values in the original and reduced dynamics, even in the limit of $\eta\to 0$.

Our main approach is to investigate the dependence of entropy production-like quantities on the levels of description. The key quantity we consider is the excess entropy production, defined as the entropy production from which the steady state housekeeping part is subtracted. There are many types of excess entropy production corresponding to the different definitions of the housekeeping part, which, in all cases, reduce to the ordinary entropy production in equilibrium dynamics, and are equal to zero at the steady state. First, we consider the excess entropy production introduced by Hatano and Sasa [12],

$$\sigma_{\mathrm{ex}}(t,\lambda_t) := -\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\langle \ln\frac{P^{\lambda_t}_{\mathrm{ss}}(\mathbf{x}_t)}{P^{\lambda_t}_{\mathrm{ss}}(\mathbf{x}_{t+\Delta t})}\right\rangle. \tag{6}$$

$P^{\lambda_t}_{\mathrm{ss}}(\mathbf{x})$ is the steady state density function under λ_t. For any Markovian stochastic dynamics [17] (originally shown in overdamped Langevin dynamics [12]), $\sigma_{\mathrm{ex}}(t,\lambda_t)$ satisfies, for an arbitrary $P_t(\mathbf{x})$, the Hatano-Sasa inequality,

$$\dot{S}[P_t(\mathbf{x})] + \sigma_{\mathrm{ex}}(t,\lambda_t) \geq 0. \tag{7}$$

This inequality may be considered as an extension of the second law to a nonequilibrium setup, the equality of which is achieved in steady state, whereas that of Eq. (5) is achieved only in equilibrium state. When the condition Eq. (3) holds, the excess entropy productions, $\sigma^{x,y}_{\mathrm{ex}}(t,\lambda_t)$ and $\sigma^{x}_{\mathrm{ex}}(t,\lambda_t)$, corresponding to two levels [Eq. (1) and Eq. (2)], are related as,

$$\begin{aligned}
\dot{S}[P_t(x,y)] + \sigma^{x,y}_{\mathrm{ex}}(t,\lambda_t) &\simeq \frac{1}{\Delta t}\left\langle \ln\frac{P_{t+\Delta t}(x_{t+\Delta t},y_{t+\Delta t})P^{\lambda_t}_{\mathrm{ss}}(x_t,y_t)}{P_t(x_t,y_t)P^{\lambda_t}_{\mathrm{ss}}(x_{t+\Delta t},y_{t+\Delta t})}\right\rangle \\
&\simeq \frac{1}{\Delta t}\left\langle \ln\frac{P_{t+\Delta t}(x_{t+\Delta t})P^{\lambda_t}_{\mathrm{ss}}(x_t)}{P_t(x_t)P^{\lambda_t}_{\mathrm{ss}}(x_{t+\Delta t})}\right\rangle + \left\langle \dot{\lambda}_t\frac{\partial}{\partial\lambda}\ln P^{\lambda}(y_t|x_t)\right\rangle \\
&\simeq \dot{S}[P_t(x)] + \sigma^{x}_{\mathrm{ex}}(t,\lambda_t).
\end{aligned} \tag{8}$$

Here, $\simeq$ indicates that the terms $O(\Delta t), O(\eta)$ are ignored, and we used $\langle\dot{\lambda}_t\frac{\partial}{\partial\lambda}\ln P^{\lambda}(y_t|x_t)\rangle = 0$. Equation (8) states that, although the excess entropy production itself may depend on the levels of description, the difference $\sigma^{x,y}_{\mathrm{ex}}(t,\lambda_t) - \sigma^{x}_{\mathrm{ex}}(t,\lambda_t)$ amounts to the contribution of the reversible heat $\frac{\mathrm{d}}{\mathrm{d}t}(S[P_t(x,y)] - S[P_t(x)])$, and therefore the Hatano-Sasa inequality (7) is kept invariant between the different levels.

Next, we investigate the excess entropy production adopted by Komatsu *et al.* [13]. They considered the steady state entropy production for a fixed parameter λ, $\sigma_{\mathrm{ss(KNST)}}(\lambda) := \sigma(t\to\infty,\lambda)$, and defined the excess entropy production as

$$\sigma_{\mathrm{ex(KNST)}}(t,\lambda_t) = \sigma(t,\lambda_t) - \sigma_{\mathrm{ss(KNST)}}(\lambda_t) \tag{9}$$

[13]. In the quasistatic parameter change from λ_0 to $\lambda_{\mathcal{T}}$ starting with the steady state $P^{\lambda_0}_{\mathrm{ss}}(\mathbf{x})$, $\sigma_{\mathrm{ex(KNST)}}(t,\lambda_t)$ satisfies the extended Clausius relation [13],

$$S_{\mathrm{sym}}[P_{\mathcal{T}}] - S_{\mathrm{sym}}[P_0] = -\int_0^{\mathcal{T}}\sigma_{\mathrm{ex(KNST)}}(t,\lambda_t)\mathrm{d}t + O(\epsilon^2\Delta), \tag{10}$$

where, $S_{\mathrm{sym}}[P_t(\mathbf{x})] = \int \mathrm{d}\mathbf{x}P_t(\mathbf{x})\frac{1}{2}(\ln P_t(\mathbf{x}) + \ln P_t(\bar{\mathbf{x}}))$ is the symmetrized Shannon entropy, ϵ and Δ are the dimensionless quantities which characterize the "degree of nonequilibrium" and the change $\lambda_{\mathcal{T}} - \lambda_0$, respectively [13]. According to the relation between $\sigma_{\mathrm{ex(KNST)}}(t,\lambda_t)$ and Hatano-Sasa-like excess entropy production [18], the sum of the Shannon entropy increment and the excess entropy production, $\dot{S}[P_t(x)] + \sigma_{\mathrm{ex(KNST)}}(t,\lambda_t)$, is kept invariant within an error of $O(\epsilon^2\Delta)$ for the case where Eq. (3) holds and $P^{\lambda_t}(y|x)$ possesses the symmetry $P^{\lambda_t}(y|x) = P^{\bar{\lambda}_t}(\bar{y}|\bar{x})$. It follows from this invariance that the extended Clausius relation (10) is invariant with respect to the change in the level of description, as

$$\begin{aligned}
S_{\mathrm{sym}}[P_{\mathcal{T}}(x,y)] - S_{\mathrm{sym}}[P_0(x,y)] + \int_0^{\mathcal{T}}\sigma^{x,y}_{\mathrm{ex(KNST)}}(t,\lambda_t)\mathrm{d}t \\
= S_{\mathrm{sym}}[P_{\mathcal{T}}(x)] - S_{\mathrm{sym}}[P_0(x)] + \int_0^{\mathcal{T}}\sigma^{x}_{\mathrm{ex(KNST)}}(t,\lambda_t)\mathrm{d}t \\
+ O(\epsilon^2\Delta) + O(\eta),
\end{aligned} \tag{11}$$

under an additional condition, $P^{\lambda_t}(y|x) = P^{\lambda_t}(\bar{y}|\bar{x})$, which is the equivalent condition for a fluctuation theorem of the hidden entropy production to hold [9].

Overdamped Langevin system.— To clarify the existence of hidden entropy production and the invariance of excess

entropy production, we investigate a two-dimensional overdamped Langevin model described by,

$$\gamma_t \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{F}_t(x, y) + \sqrt{2\gamma_t T_t} \begin{pmatrix} \xi_x \\ \xi_y \end{pmatrix}. \tag{12}$$

Here, index t represents time-dependence of the external parameters included in λ_t, γ_t is the drag coefficient, T_t is the temperature of the environment, ξ_x, ξ_y are independent white Gaussian noises with unit variance. The general external force $\mathbf{F}_t(x, y)$ may be written by the scalar potential $U_t(x, y; \alpha)$ and the vector potential parallel to the z-direction $\mathbf{V}_t(x, y; \alpha) = V_t(x, y; \alpha)\,\mathbf{e}_z$ as $\mathbf{F}_t = -\nabla U_t - \nabla \times \mathbf{V}_t$. The nonconservative force, $-\nabla \times \mathbf{V}_t$, drives the system out of equilibrium.

The separation of time scale and the strength of the nonconservative force in Eq. (12) are controlled by introducing the parameter α as

$$U_t(x, y; \alpha) = \check{U}_t(x, \alpha y), \quad V_t(x, y; \alpha) = \alpha^{-1} \check{V}_t(x, \alpha y). \tag{13}$$

As α becomes large, the scalar potential $U_t(x, y; \alpha)$ squeezes the extent of the probability density along the y-direction, and the diffusive time scale, $\tau_y \propto \alpha^{-2}$, becomes small compared with the other time scales, τ_x and τ_λ. By the procedure of singular perturbation theory [19], an arbitrary probability density function given at time $t = 0$ converges to

$$P_t(x, y) = P_t(x) \frac{\exp\left(-U_t(x, y; \alpha)/T_t\right)}{Z_t(x)} + O(\eta), \tag{14}$$

at $t \gg \tau_y$, and consequently Eq. (12) is reduced to the closed Langevin dynamics of x,

$$\gamma_t \dot{x} = \left(T_t \frac{\partial}{\partial x} \ln Z_t(x) + f_t^V(x) + \sqrt{2\gamma_t T_t} \xi_x \right) [1 + O(\eta)]. \tag{15}$$

Derivation is given at the end of the Letter. $Z_t(x)$ is the local partition function,

$$Z_t(x) = \alpha \int \exp\left(-\frac{U_t(x, y; \alpha)}{T_t} \right) \mathrm{d}y, \tag{16}$$

and $f_t^V(x)$ is the averaged x-component of the force that originates from the vector potential,

$$f_t^V(x) = \frac{1}{Z_t(x)} \int \left(-\frac{\partial V_t(x, y; \alpha)}{\partial y} \right) \exp\left(-\frac{U_t(x, y; \alpha)}{T_t} \right) \mathrm{d}y. \tag{17}$$

Note that the Langevin equation (15) at the reduced level may have an equilibrium state even when Eq. (12) at the original level has no equilibrium state in the limit $\eta \to 0$.

We examine the total entropy production at the original level [Eq. (12)] and the reduced level [Eq. (15)]. For a Langevin system, the entropy production at each level can be rewritten as,

$$\sigma^{x,y}(t, \lambda_t) = \frac{1}{T_t} \left\langle - \left[-\gamma_t \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} + \sqrt{2\gamma_t T_t} \begin{pmatrix} \xi_x \\ \xi_y \end{pmatrix} \right] \circ \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \right\rangle, \tag{18}$$

$$\sigma^{x}(t, \lambda_t) = \frac{1}{T_t} \left\langle -[-\gamma_t \dot{x} + \sqrt{2\gamma_t T_t} \xi_x] \circ \dot{x} \right\rangle \tag{19}$$

[2], where $\circ$ denotes the product in the Stratonovich sense. By substituting Eqs. (12), (15) into Eqs. (18), (19) and evaluating the average using the local stationary density (14), we may show

$$\begin{aligned} \sigma^{x,y}(t, \lambda_t) - \sigma^{x}(t, \lambda_t) = &-\frac{\mathrm{d}}{\mathrm{d}t} \left\langle \frac{U_t(x, y; \alpha)}{T_t} - \ln Z_t(x) \right\rangle \\ &+ \frac{1}{\gamma_t T_t} \left\langle \left(\frac{\partial V}{\partial y} \right)^2 - f_v(x)^2 \right\rangle [1 + O(\eta)]. \end{aligned} \tag{20}$$

The first term of the right hand side is equivalent to the reversible heat, $\frac{\mathrm{d}}{\mathrm{d}t}(S[P_t(x, y)] - S[P_t(x)])$, therefore the hidden entropy production is given by,

$$\begin{aligned} \Xi(t, \lambda_t) &:= \Sigma^{x,y}(t, \lambda_t) - \Sigma^{x}(t, \lambda_t) \\ &= \left(\frac{1}{\gamma_t T_t} \left\langle \left(\frac{\partial V}{\partial y} \right)^2 - \left(f_t^V(x) \right)^2 \right\rangle \right) [1 + O(\eta)], \end{aligned} \tag{21}$$

which is a non-negative quantity since the right hand side of Eq. (21) is the variance of $\frac{\partial V(x, y; \alpha)}{\partial y}$,

Let us describe here the simplest example, $U(x, y; \alpha) = u(x) + k(\alpha y)^2/2$ and $V(x, y; \alpha) = (Ik/T)^{\frac{1}{2}} (\alpha y)^2/\alpha$. The dynamics of x is described by,

$$\gamma \dot{x} = -u'(x) + \zeta(t) + \sqrt{2\gamma T} \xi_x, \tag{22}$$

where $\zeta = -(Ik/T)^{\frac{1}{2}} \alpha y$ may be considered as a fluctuating external force, which behaves as a colored Gaussian noise with correlation time $\tau_y = \gamma/k\alpha^2$, $\langle \zeta(t)\zeta(s) \rangle = I \exp\left(-|t - s|/\tau_y\right)$. In this (x, y)-level, the entropy production is obtained as,

$$\sigma^{x,y}(t) = \gamma^{-1} \langle (-u'(x) + \zeta) \circ \dot{x} \rangle = \gamma^{-1} (\langle -u'(x) \circ \dot{x} \rangle + I), \tag{23}$$

in the limit of $\tau_y \to 0$. Now ζ in Eq. (22) has no effect on the dynamics of x in this limit, since it has no singularity like that in ξ_x. Therefore, y may be eliminated from the description of dynamics, and $\gamma \dot{x} = -u'(x) + \sqrt{2\gamma T} \xi_x$ is obtained. Entropy production in this x-level is,

$$\sigma^{x}(t) = \gamma^{-1} \langle -u'(x) \circ \dot{x} \rangle. \tag{24}$$

The discrepancy, $\gamma^{-1} I$, between Eqs. (23) and (24) corresponds to the hidden entropy production, $\Xi(t, \lambda_t)$. This example demonstrates that a *hidden force* like ζ which may be present in any setup affects the value of entropy production, indicating that the standard method of stochastic energetics fails to evaluate the entropy production in a level-independent manner.

We have seen that the hidden entropy production in our model is generally positive, even in the limit of $\eta \to 0$, where the reduction of dynamics to Eq. (15) is completely justified. In contrast, the excess entropy productions are kept invariant between the levels, since Eq. (14) meets the condition (3) in the previous general analysis. The invariance of the extended Clausius relation (10) may also be directly verified by an order

estimation of the excess part derived from the hidden entropy production,

$$\int_0^{\mathcal{T}} [\Xi(t,\lambda_t) - \Xi(t' \to \infty, \lambda_t)]\,\mathrm{d}t \sim \frac{l_x^2 \Delta}{T_t^2} \left\langle \left(\frac{\partial V_t}{\partial y}\right)^2 - (f_t^V)^2 \right\rangle \sim \epsilon^2 \Delta. \tag{25}$$

Here, we introduced the typical length scale along the x-direction l_x, to relate the degree of nonequilibrium $\epsilon := \frac{l_x}{T}\frac{\partial V}{\partial y}$ and the deviation of the probability density distribution from the steady distribution, which is proportional to τ_x/τ_λ with $\tau_x = O(l_x^2 \gamma_T / T_t)$ and $\tau_\lambda = O(\mathcal{T}/\Delta)$.

In this model, another definition of the excess entropy production is also kept invariant between the levels. For an arbitrary overdamped Langevin system, Maes and Netočný found that for fixed parameters except U, a given distribution P_t is a steady state of the dynamics with the potential $U = \mathrm{argmin}_U \Sigma^{x,y}(t,\lambda_t)$. Based on this fact, they defined the steady entropy production as $\sigma_{\mathrm{ss(MN)}}(t,\lambda_t) := \min_U \Sigma^{x,y}(t,\lambda_t)$ [14]. Then, their modified excess entropy production is defined as

$$\sigma_{\mathrm{mex}}(t,\lambda_t) = \sigma(t,\lambda_t) - \sigma_{\mathrm{ss(MN)}}(t,\lambda_t), \tag{26}$$

which satisfies another second law-like inequality for an arbitrary $P_t(\mathbf{x})$,

$$\dot{S}[P_t(\mathbf{x})] + \sigma_{\mathrm{mex}}(t,\lambda_t) \geq 0, \tag{27}$$

in the overdamped Langevin system [14]. It follows from Eq. (21) that the left hand sides of the inequality (27) in the original level and the reduced level are equivalent in the limit of small η,

$$\dot{S}[P_t(x,y)] + \sigma^{x,y}_{\mathrm{mex}}(t,\lambda_t) = \left(\dot{S}[P_t(x)] + \sigma^{x}_{\mathrm{mex}}(t,\lambda_t)\right)[1 + O(\eta)]. \tag{28}$$

These invariances of the excess entropy productions [Eqs. (8), (11), (28)] indicate that the hidden entropy production (21) is essentially included in the housekeeping part, not in the excess part.

Remarks.— In the present Letter, we analyzed the two-dimensional overdamped Langevin model to exemplify the invariance of the excess entropy production. Similar structures, of course, exist in other systems; for example, underdamped Langevin systems that can be reduced to overdamped Langevin dynamics. As originally pointed out in [8], spatially inhomogeneous temperature produces difference in the total entropy production between the underdamped description and the overdamped description (general expression was given by [20]). Since Eq. (3) is valid in the overdamped limit, we can show that the Hatano-Sasa inequality (7) [12] and the extended Clausius relation (10) [13] are kept invariant between the two descriptions.

We further comment on the relation between the ordinary entropy production and the excess entropy production. In the special case where the reduced dynamics has an equilibrium state, the excess entropy productions are equivalent to the ordinary entropy production at the reduced level, $\sigma^x(t,\lambda_t) = \sigma^x_{\mathrm{ex}}(t,\lambda_t) = \sigma^x_{\mathrm{mex}}(t,\lambda_t) = \sigma^x_{\mathrm{ex(KNST)}}(t,\lambda_t)$. As we have shown, these entropy productions are also equivalent to $\sigma^{x,y}_{\mathrm{ex}}(t,\lambda_t)$ and $\sigma^{x,y}_{\mathrm{mex}}(t,\lambda_t)$. Therefore, the nonequilibrium second law-like inequalities [(7) or (27)] in the original level are equivalent to the inequality for the total entropy production at the reduced level, $\Sigma^x(t,\lambda_t) \geq 0$, unlike the corresponding inequality at the original level $\Sigma^{x,y}(t,\lambda_t) \geq 0$.

Conclusion.— We considered general Markovian stochastic processes which can be described by two levels of description, and showed that the excess entropy productions are essentially invariant with respect to the change in the levels, even in the case where the hidden entropy production exists. These results were investigated in the two-dimensional overdamped Langevin model.

Our results are encouraging for the experimental investigation of SST. Measurable variables are often restricted to few degrees of freedom, due to the limitation in experimental techniques. In such cases, as we have verified, the total entropy production does not generally describe the physical laws in a level-independent manner. It is expected, however, that the excess entropy production possesses an objective value, when the dynamics is closed (in the Markovian sense) among the measurable quantities. Therefore, the objectivity of the physical laws is recovered by considering the excess entropy production, supporting the possibility of experimentally exploring SST based on the mesoscopic level of description.

Derivations.— We derive the reduced Langevin equation (15) using the multi-scale analysis. The Fokker-Planck equation corresponding to Eq. (12) may be written by the rescaled variable $Y = \alpha y$,

$$\begin{aligned} \partial_t P_t = &-\gamma^{-1}\partial_x\left(\left(-\partial_x \check{U} - \partial_Y \check{V}\right)P_t - T\partial_x P_t\right) - \gamma^{-1}\partial_Y\left((\partial_x \check{V})P_t\right) \\ &- \alpha^2\gamma^{-1}\partial_Y\left(-(\partial_Y \check{U})P_t - T\partial_Y P_t\right). \end{aligned} \tag{29}$$

Since the first line of the right hand side of Eq. (29) is the order of $O(\eta)$ as compared with the second line, the first line can be treated as the perturbative terms in the limit of $\eta \to 0$. According to the multi-scale analysis, two dimensionless times, $t_s = \eta t/\tau_y$ and $t_y = t/\tau_y$, are introduced, and the probability density function is expanded as,

$$P_t(x,Y) = P^{(0)}_{t_s,t_y}(x,Y) + O(\eta). \tag{30}$$

$P^{(0)}_{t_s,t_y}$ obeys the leading order equation in Eq. (29),

$$\tau_y^{-1}\partial_{t_y} P^{(0)}_{t_s,t_y} = -\alpha^2\gamma^{-1}\partial_Y\left(-(\partial_Y \check{U})P^{(0)}_{t_s,t_y} - T\partial_Y P^{(0)}_{t_s,t_y}\right), \tag{31}$$

and converges to a local stationary density, $P^{(0)}_t = P^{(0)}_t(x)\frac{1}{Z(x)}\exp\left(-\check{U}/T\right)$ in a time scale longer than τ_y [Eq. (14)], where $P^{(0)}_t(x) = \int P^{(0)}_t(x,Y)\mathrm{d}Y$. In such time scale, the equation obtained from the integral over y in the next order,

$$\begin{aligned} &\tau_x^{-1}\left(\partial_{t_s} P^{(0)}_{t_s,t_y}(x) + \partial_{t_y} P^{(1)}_{t_s,t_y}(x)\right) \\ &\quad = -\gamma^{-1}\partial_x\left(\partial_x T\ln Z(x) + f^V_t(x) - T\partial_x\right)P^{(0)}_{t_s,t_y}(x), \end{aligned} \tag{32}$$

gives the Fokker-Planck equation equivalent to Eq. (15) in the limit of $\eta \to 0$, since the left hand side of Eq. (32) is $\partial_t P_t(x)(1 + O(\eta))$.

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